

# From Alternating Sign Matrices to the Gaussian Unitary Ensemble

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**Abstract:** The aim of this note is to prove that fluctuations of uniformly random alternating sign matrices (equivalently, configurations of the 6-vertex model with domain wall boundary conditions) near the boundary are described by the Gaussian Unitary Ensemble and the GUE-corners process.

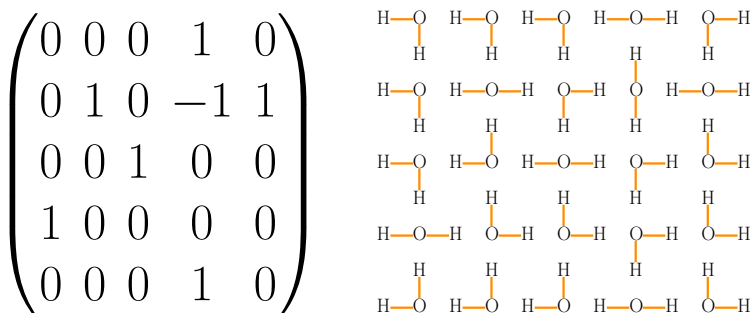
## 1. Introduction

An *Alternating Sign Matrix* (ASM) of size  $N$  is a  $N \times N$  matrix whose entries are either  $0$ ,  $1$ , or  $-1$ , such that the sum along every row and column is  $1$  and, moreover, along each row and each column the nonzero entries alternate in sign, see Fig. 1 for an example.

Since their introduction by Mills–Robbins–Rumsey [MRR] ASMs attracted lots of attention both in combinatorics and in mathematical physics. Enumerative properties of ASMs show their deep connections with various classes of plane partitions and with a number of well-known lattice models, see e.g., recent reviews in [Z2, G], [BFZ, Introduction] and references therein. Great interest in ASMs in statistical mechanics is related to the fact that they are in bijection with configurations of the 6-vertex model (or with square ice model) with domain-wall boundary conditions as shown at Fig. 1. A good review of the 6-vertex model can be found, e.g., in the book [Bax] by Baxter.

Our interest in ASMs is probabilistic. We would like to know how a *uniformly random* ASM of size  $N$  looks like when  $N$  is large. The features of this model are believed to be similar to the *dimer models*, i.e., random lozenge tilings, plane partitions and domino tilings, cf. [Ke] and also [EKLP, CLP, J, KOS, BGR, P]. However, one of the key tools for studying the dimer models is the fact that they can be described via *determinantal point processes*. Such structure is not known for uniformly random ASMs and one has to find different methods.

One of the (conjectural) features of uniformly random ASMs is the formation of the so-called limit shape (also present in the dimer models), whose properties were studied



**Fig. 1.** An alternating sign matrix of size 5 and the corresponding configuration of the 6-vertex model (*square ice*) with domain wall boundary condition. 1s in ASM correspond to horizontal molecules H–O–H and –1s to the vertical ones

by Colomo and Pronko [CP]; for the 6-vertex model with more general boundary conditions the limit shape phenomenon is discussed in [PR,R] (see also [Z1]). For ASMs the limit shape theorem would claim, in particular, that when  $N$  is large all non-zero matrix entries of a uniformly random ASM of size  $N$  lie with high probability inside a certain deterministic curve, inscribed in  $N \times N$  rectangle, see [CP] for the details. As far as the author knows, the exact form of this curve is still conjectural, but it closely matches the numeric simulations of [AR,SZ].

Continuing the conjectural analogy with the dimer models, one expects various connections with random matrices. In this article we study the asymptotic fluctuations of ASMs near the boundary of the square and find such connection, which we now present.

Recall that the Gaussian Unitary Ensemble (GUE) of rank  $N$  is the ensemble of random Hermitian matrices  $X = \{X_{ij}\}_{i,j=1}^N$  with probability density (proportional to)  $\exp(-\text{Trace}(X^2/2))$  with respect to the Lebesgue measure. Let  $\lambda_1^N \leq \lambda_2^N \leq \dots \leq \lambda_N^N$  denote the eigenvalues of  $X$  and, more generally, for  $1 \leq k \leq N$  let  $\lambda_1^k \leq \lambda_2^k \leq \dots \leq \lambda_k^k$  denote the eigenvalues of top-left  $k \times k$  corner  $\{X_{ij}\}_{i,j=1}^k$  of  $X$ . The joint distribution of  $\lambda_i^j, i = 1, \dots, j, j = 1, \dots, N$  is known as the *GUE-corners* process of rank  $N$  (the name GUE-minors process is also used, cf. [JN]). The following theorem is the main result of the present article.

**Theorem 1.** Fix any  $k$ .

1. As  $N \rightarrow \infty$  the probability that the number of –1s in the first  $k$  rows of a uniformly random ASM of size  $N$  is maximal possible (i.e., there is one –1 in the second row, two –1s in the third row, etc) tends to 1, and, thus, there are  $k(k-1)/2$  interlacing 1s in the first  $k$  rows with high probability.
2. Let  $\eta(N)_j^i, i = 1, \dots, j, j = 1, \dots, k$  denote the column number of the  $i$ th 1 in the  $j$ th row of the uniformly random ASM, where we agree that  $\eta(N)_j^i = +\infty$  if there are less than  $i$  1s in the  $j$ th row. Then the random vector

$$\sqrt{\frac{8}{3N}} \left( \eta(N)_j^i - N/2 \right) \quad (1)$$

weakly converges to the GUE-corners process as  $N \rightarrow \infty$ .

*Remark.* Symmetries of uniformly random ASMs imply an analogue of Theorem 1 for the last  $k$  rows, first  $k$  columns and last  $k$  columns of ASM. It is very plausible that the four limiting GUE-corners processes are jointly independent.

A number of results similar to Theorem 1 for models of random Young diagrams and random tilings related to the determinantal point processes is known, see [Bar, JN, OR, No, GS, GP]. Moreover, for random lozenge tilings the GUE-corners process is believed to be the *universal scaling limit* near an edge of the boundary of the tiled domain, cf. [OR, JN, GP]. Interestingly, the number of ASMs is the same as the number of lozenge tilings of a hexagon with certain symmetries (see e.g., [BP] and references therein). However, this fact remains quite mysterious and no bijective proof of it is known; Theorem 1, thus, gives another indication that direct combinatorial connection between ASMs and lozenge tilings should exist.

Theorem 1 was conjectured in [GP], in the same paper a partial result towards Theorem 1 was proved. Our argument relies on this result, so let us present it.

Let  $\Psi_k(N)$  denote the sum of coordinates of  $\mathbf{1}$ s minus the sum of coordinates of  $-\mathbf{1}$ s in the  $k^{\text{th}}$  row of the uniformly random ASM of size  $N$ . In [GP] it is proved that the centered and rescaled random variables  $\Psi_k(N)$  converge to the collection of i.i.d. Gaussian random variables as  $N \rightarrow \infty$ .

**Theorem 2** (Theorem 1.10 in [GP]). *For any fixed  $k$  the random variable  $\sqrt{\frac{8}{3}} \frac{\Psi_k(N) - N/2}{\sqrt{N}}$  weakly converges to the standard normal random variable  $N(0, 1)$ . Moreover, the joint distribution of any collection of such variables converges to the distribution of independent standard normal random variables.*

We believe (but we do not have a proof) that an analogue of Theorem 1 should hold for more general measures on ASMs. A natural class of measures can be obtained through the correspondence with 6-vertex model. In the latter model one typically subdivides six types of vertices into three groups and assigns weights  $a, b, c$  to these three groups. The probability of a configuration is further set to be proportional to the product of the weights of its vertices. For instance, these are the settings of the celebrated Izergin–Korpein formula [I, Kor] for the partition function of the 6-vertex model with domain wall boundary conditions. Asymptotics of this partition function in the limit regime which is somewhat similar to the one used in arguments of [GP] (leading to Theorem 2) was also investigated in [CP2, Appendix B], [CPZ, Appendix].

For one particular choice of the parameters  $a, b$  and  $c$  known as “the free fermion point” of the 6-vertex model, an analogue of Theorem 1 follows from the results of [JN]. In terms of the ASMs this choice of weights corresponds to assigning the probability proportional to  $2^{n_1}$  to an alternating sign matrix with  $n_1$   $\mathbf{1}$ s. This case is closely related to uniformly random domino tilings of the Aztec diamond (as is explained in [EKLP, FS]), to Schur measures (see [BG] for a recent review) and to determinantal point processes, which makes it somewhat simpler.

In the rest of the article we provide a proof of Theorem 1, which is organized as follows. In Section 2 we study various classes of Gelfand–Tsetlin patterns and Gibbs measures on them. In Sect. 3 we prove that the distribution of random vector (1) is tight as  $N \rightarrow \infty$ . In Sect. 4 we combine all the obtained results to finish the proof.

## 2. Gibbs Measures on Gelfand–Tsetlin Patterns

**2.1. Half-Strict Gelfand–Tsetlin patterns.** Let  $\mathbb{GT}_N$  denote the set of  $N$ -tuples of *distinct* integers:

$$\mathbb{GT}_N = \{\lambda \in \mathbb{Z}^N \mid \lambda_1 < \lambda_2 < \cdots < \lambda_N\}. \quad (2)$$

We say that  $\lambda \in \mathbb{GT}_N$  and  $\mu \in \mathbb{GT}_{N-1}$  *interlace* and write  $\mu \prec \lambda$  if

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{N-1} \leq \lambda_N. \quad (3)$$

Note that the inequalities in (2) are strict, while in (3) they are weak.

Let  $\mathbb{GT}^{(N)}$  denote the set of sequences

$$\mu^1 \prec \mu^2 \prec \cdots \prec \mu^N, \quad \mu^i \in \mathbb{GT}_i, \quad 1 \leq i \leq N, \quad \mu^i \prec \mu^{i+1}, \quad 1 \leq i < N.$$

We call the elements of  $\mathbb{GT}^{(N)}$  *half-strict Gelfand–Tsetlin patterns*<sup>1</sup> (they are also known as monotonous triangles, cf. [MRR]).

For  $\lambda \in \mathbb{GT}_N$ , let  $\mathbb{GT}_\lambda^{(N)} \subset \mathbb{GT}^{(N)}$  denote the set of half-strict Gelfand–Tsetlin patterns  $\mu^1 \prec \cdots \prec \mu^N$  such that  $\mu^N = \lambda$ .

**Lemma 3.** *The set of ASMs of size  $N$  is in bijection with  $\mathbb{GT}_{1 < 2 < \cdots < N}^{(N)}$ . The bijection is given by*

$$ASM = (r^1, \dots, r^N) \mapsto \mu^1 \prec \mu^2 \prec \cdots \prec \mu^N \in \mathbb{GT}_{1 < 2 < \cdots < N}^{(N)},$$

where  $\mu^k$  encodes the column numbers of **1**s in the sum of the first  $k$  rows  $r^1 + \cdots + r^k$  of an ASM.

*Proof.* This is straightforward, see also [MRR].  $\square$

Under the above identification, the random variables  $\Psi_k(N)$  of Theorem 2 turn into the differences

$$\Psi_k(N) = |\mu^k| - |\mu^{k-1}|,$$

where  $|\mu^k|$  is the sum of coordinates  $\mu_1^k + \cdots + \mu_k^k$  of  $\mu^k \in \mathbb{GT}_N$ , and  $\mu^1 \prec \mu^2 \prec \cdots \prec \mu^N$  is the uniformly random element of  $\mathbb{GT}_{1 < 2 < \cdots < N}^{(N)}$ .

**Definition 4.** A probability measure  $\rho$  on  $\mathbb{GT}^{(k)}$  is called *Gibbs measure* if for any  $\lambda \in \mathbb{GT}_k$ , the restriction of  $\rho$  on  $\mathbb{GT}_\lambda^{(k)}$  is proportional to the uniform distribution on  $\mathbb{GT}_\lambda^{(k)}$ :

$$\rho \Big|_{\mathbb{GT}_\lambda^{(k)}} = \rho_k(\lambda) \cdot \text{Uniform measure on } \mathbb{GT}_\lambda^{(k)},$$

where  $\rho_k(\cdot)$  is the projection of  $\rho$  on  $\mathbb{GT}_k$ .

Clearly, if  $\mu^1 \prec \mu^2 \prec \cdots \prec \mu^N \in \mathbb{GT}_{1 < 2 < \cdots < N}^{(N)}$  corresponds to uniformly random ASM as in Lemma 3, then for any  $1 \leq k \leq N$ , the distribution of  $\mu^1 \prec \mu^2 \prec \cdots \prec \mu^k$  is a Gibbs measure on  $\mathbb{GT}^{(k)}$ .

<sup>1</sup> The name comes from the fact that an analogous object when all the inequalities are not strict is closely related to the representations of unitary groups and Gelfand–Tsetlin basis in such irreducible representations.

**2.2. Continuous Gibbs property.** Let us introduce a continuous analogue of the set of half-strict Gelfand–Tsetlin patterns  $\widehat{\mathbb{GT}}^{(N)}$ .

Let  $\widehat{\mathbb{GT}}_N$  denote the set of  $N$ -tuples of reals:

$$\widehat{\mathbb{GT}}_N = \{\lambda \in \mathbb{R}^N \mid \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N\}. \quad (4)$$

We say that  $\lambda \in \widehat{\mathbb{GT}}_N$  and  $\mu \in \widehat{\mathbb{GT}}_{N-1}$  *interlace* and write  $\mu \prec \lambda$  if

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{N-1} \leq \lambda_N. \quad (5)$$

Let  $\widehat{\mathbb{GT}}^{(N)}$  denote the set of sequences

$$\mu^1 \prec \mu^2 \prec \cdots \prec \mu^N, \quad \mu^i \in \widehat{\mathbb{GT}}_i, \quad 1 \leq i \leq N, \quad \mu^i \prec \mu^{i+1}, \quad 1 \leq i < N.$$

We call the elements of  $\widehat{\mathbb{GT}}^{(N)}$  *continuous Gelfand–Tsetlin patterns*.

For  $\lambda \in \widehat{\mathbb{GT}}_N$ , let  $\widehat{\mathbb{GT}}_\lambda^{(N)} \subset \widehat{\mathbb{GT}}^{(N)}$  denote the set of continuous Gelfand–Tsetlin patterns  $\mu^1 \prec \cdots \prec \mu^N$  such that  $\mu^N = \lambda$ .

The following definition is a straightforward analogue of Definition 4.

**Definition 5.** A probability measure  $\rho$  on  $\widehat{\mathbb{GT}}^{(k)}$  is called *Gibbs measure* if for any  $\lambda \in \widehat{\mathbb{GT}}_N$ , the conditional distribution of  $\rho$ , given that  $\mu^N = \lambda$  is the uniform distribution on  $\widehat{\mathbb{GT}}_\lambda^{(k)}$ , i.e.,

$$\rho(\cdot \mid \mu^k = \lambda) = \text{Uniform measure on } \widehat{\mathbb{GT}}_\lambda^{(k)}.$$

For  $N$ -tuple  $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N) \in \widehat{\mathbb{GT}}_N$  set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N.$$

**Proposition 6.** Let  $\rho$  be a Gibbs measure on  $\widehat{\mathbb{GT}}^{(N)}$  and let  $\mu^1 \prec \mu^2 \prec \cdots \prec \mu^N$  be  $\rho$ -distributed random element of  $\widehat{\mathbb{GT}}^{(N)}$ . Suppose that

$$|\mu^1|, |\mu^2| - |\mu^1|, |\mu^3| - |\mu^2|, \dots, |\mu^N| - |\mu^{N-1}|$$

is a Gaussian vector with i.i.d.  $N(0, 1)$ -distributed components. Then  $\rho$  is the GUE-corners process of rank  $N$ .

*Proof.* Let  $\mathcal{H}(N)$  denote the set of  $N \times N$  Hermitian matrices and let  $U(N)$  denote the group of all  $N \times N$  unitary matrices. Note that  $U(N)$  acts on  $\mathcal{H}(N)$  by conjugations and this action preserves eigenvalues of Hermitian matrices. Take any  $\lambda \in \widehat{\mathbb{GT}}_N$ , let  $X(\lambda)$  denote the diagonal matrix with eigenvalues  $\lambda_1, \dots, \lambda_N$  and let  $\mathbb{O}_\lambda$  denote the  $U(N)$ -orbit of  $X(\lambda)$ . Further, let  $\mathcal{O}_\lambda$  denote the *orbital measure* on  $\mathbb{O}_\lambda$ , which is the pushforward of the (normalized) Haar measure on  $U(N)$  with respect to the map

$$U(N) \rightarrow \mathcal{H}(N), \quad u \mapsto uX(\lambda)u^{-1}.$$

Equivalently, if we view  $\mathcal{H}(N)$  as the real Euclidian space of dimension  $N^2$  equipped with norm  $\|X\|^2 = \text{Trace}(X^2)$ , then  $\mathcal{O}_\lambda$  is merely a uniform measure on the orbit  $\mathbb{O}_\lambda$ .

Now let  $\mu^1 \prec \mu^2 \prec \cdots \prec \mu^N$  be distributed according to  $\rho$  and let  $\rho_N$  denote the measure on  $\widehat{\mathbb{GT}}_N$  which is the projection of  $\rho$  on  $\mu^N$ .

Further let  $\Theta_\rho$  denote the  $U(N)$ -invariant measure on  $\mathcal{H}(N)$  which is  $\rho_N$  mixture of the orbital measures  $\mathcal{O}_\lambda$ . In other words, for any Borel set  $\mathcal{A} \subset \mathcal{H}(N)$  we set

$$\Theta_\rho(\mathcal{A}) = \int_{\widehat{\mathbb{GT}}_N} \mathcal{O}_\lambda(A) \rho_N(d\lambda).$$

Suppose that  $M = \{M_{ij}\}_{i,j=1}^N$  is a random  $\Theta_\rho$ -distributed Hermitian matrix. Define  $v^k \in \widehat{\mathbb{GT}}_k$ ,  $k = 1, \dots, N$ , to be the eigenvalues of top-left  $k \times k$  corner of  $M$ , i.e., of  $\{M_{ij}\}_{i,j=1}^k$ . Straightforward linear algebra shows that

$$v^1 \prec v^2 \prec \dots \prec v^N.$$

We claim that the distribution of the vector  $(v^k)$ ,  $1 \leq k \leq N$  is the same as that of  $(\mu^k)$ ,  $1 \leq k \leq N$ . Indeed, the distributions of  $\mu^N$  and  $v^N$  coincide by the construction. The conditional distribution of  $(\mu^k)$ ,  $1 \leq k \leq N-1$  given  $\mu^N$  is uniform, since  $\rho$  is a Gibbs measure. The distribution of  $(v^k)$ ,  $1 \leq k \leq N-1$  given  $v^N$  is also uniform, which is a known property of orbital measures  $\mathcal{O}_\lambda$ , see [GN], [Bar, Proposition 4.7], [Ne, Proposition 1.1].

Now it remains to prove that  $\Theta_\rho$  is GUE-distribution, i.e., its density with respect to Lebesgue measure is proportional to  $\exp(-\text{Trace}(X^2/2))$ . This is what we do in the rest of the proof.

Note that for  $1 \leq k \leq N$  we have

$$M_{kk} = \text{Trace}\left(\{M_{ij}\}_{i,j=1}^k\right) - \text{Trace}\left(\{M_{ij}\}_{i,j=1}^{k-1}\right) = |v^k| - |v^{k-1}|.$$

Therefore,  $M_{kk}$  are i.i.d. standard Gaussians.

Further, the distribution of  $M$  is uniquely defined by its Fourier transform  $\phi$  (i.e., characteristic function), which is

$$\phi : \mathcal{H}(N) \rightarrow \mathbb{C}, \quad \phi(A) = \mathbb{E}(\exp(\mathbf{i} \cdot \text{Trace}(AM))).$$

Suppose that a  $N \times N$  Hermitian matrix  $A$  has eigenvalues  $a_1 \leq a_2 \leq \dots \leq a_N$  and let  $\text{diag}(A)$  denote the diagonal matrix with the same eigenvalues, i.e.,  $\text{diag}(A)_{ij} = \delta_{ij}a_i$ ,  $1 \leq i, j \leq N$ . There exists  $u \in U(N)$  such that  $A = u \text{diag}(A) u^{-1}$ . Using  $U(N)$ -invariance of the distribution of  $M$  and the fact that  $\text{Trace}(uBu^{-1}) = \text{Trace}(B)$  for any matrix  $B$ , we get

$$\begin{aligned} \mathbb{E} \exp(\mathbf{i} \text{Trace}(AM)) &= \mathbb{E} \exp(\mathbf{i} \text{Trace}(u \text{diag}(A) u^{-1} \cdot u M u^{-1})) \\ &= \mathbb{E} \exp(\mathbf{i} \text{Trace}(\text{diag}(A) M)) = \mathbb{E} \exp\left(\mathbf{i} \sum_{i=1}^N a_i M_{ii}\right) = \prod_{i=1}^N \exp\left(-\frac{(a_i)^2}{2}\right), \end{aligned} \quad (6)$$

where the last equality is the computation of the Fourier transform of the Gaussian distribution. It remains to note that for the GUE-distribution, the Fourier transform is the same as the one given by (6).  $\square$

### 3. Tightness

The aim of this section is to prove the following tightness statement.

**Proposition 7.** *For  $N = 1, 2, \dots$ , let  $\xi(N) = (\xi(N)^1 < \xi(N)^2 < \dots < \xi(N)^N)$  be the uniformly random element of  $\mathbb{GT}_{(1 < 2 < \dots < N)}^{(N)}$ . Then for any  $k \geq 1$  the sequence of random variables  $N^{-1/2}(\xi(N)^k - N/2)$ ,  $N = 1, 2, \dots$  is tight (here  $\xi(N)^k$  is the index, not power).*

The proof of Proposition 7 is based on the following Lemma.

**Lemma 8.** *Fix  $N > 0$  and take a large enough positive number  $L$ . Let  $\lambda \in \mathbb{GT}_N$  be such that  $\lambda_N - \lambda_1 = L$ . Further suppose that  $\mu^1 < \dots < \mu^N$  is distributed according to the uniform measure on  $\mathbb{GT}_\lambda^{(N)}$ . Then for any  $c \in \mathbb{R}$ , we have*

$$\text{Prob} \left( |\mu_1^1 - c| > \frac{L}{2N!} \right) \geq 2^{-N-1}, \quad (7)$$

Let us first use Lemma 8 to prove Proposition 7.

*Proof of Proposition 7.* We argue by the contradiction.

Suppose that random variables  $N^{-1/2}(\xi(N)^k - N/2)$ ,  $N = 1, 2, \dots$  are not tight as  $N \rightarrow \infty$ . Since any family of bounded random variables on  $\mathbb{R}^k$  is tight, this would imply that there exist a positive number  $p > 0$ , a sequence of integers  $N_1 < N_2 < N_3 < \dots$  and a growing to  $+\infty$  sequence  $L_i$ ,  $i = 1, 2, \dots$ , such that

$$\text{Prob} \left( \sup_{j=1, \dots, k} \left| N_i^{-1/2}(\xi(N_i)_j^k - N_i/2) \right| > L_i \right) > p$$

for every  $i = 1, 2, 3, \dots$ . Since  $\xi(N_i)_1^k < \xi(N_i)_2^k < \dots < \xi(N_i)_k^k$ , one of the following three inequalities should then hold for infinitely many  $i$ s

- (I)  $\text{Prob} \left( N_i^{-1/2}(\xi(N_i)_1^k - N_i/2) > L_i/2 \right) > p/3,$
- (II)  $\text{Prob} \left( N_i^{-1/2}(\xi(N_i)_k^k - N_i/2) < -L_i/2 \right) > p/3,$
- (III)  $\text{Prob} \left( N_i^{-1/2}(\xi(N_i)_k^k - \xi(N_i)_1^k) > L_i/2 \right) > p/3.$

In case (I), due to interlacing conditions,  $\text{Prob} \left( N_i^{-1/2}(\xi(N_i)_1^1 - N_i/2) > L_i/2 \right) > p/3$ , which contradicts the convergence of  $N_i^{-1/2}(\xi(N_i)_1^1 - N_i/2)$  to a Gaussian random variable, which is proved in Theorem 2. Similarly, in case (II),  $\text{Prob} \left( N_i^{-1/2}(\xi(N_i)_1^1 - N_i/2) < -L_i/2 \right) > p/3$ , which again contradicts Theorem 2.

In case (III) we note that the conditional distribution of  $\xi(N_i)_b^a$ ,  $b = 1, \dots, a$ ,  $a = 1, \dots, k-1$  given  $\xi(N_i)^k = \lambda$  is the uniform measure on the set  $\mathbb{GT}_\lambda^{(N)}$ . Then we can use Lemma 8 and conclude that

$$\text{Prob} \left( N_i^{-1/2} |\xi(N_i)_1^1 - N_i/2| > \frac{L_i}{4k!} \right) \geq \frac{p}{3 \cdot 2^{k+1}},$$

which yet again contradicts Theorem 2.  $\square$

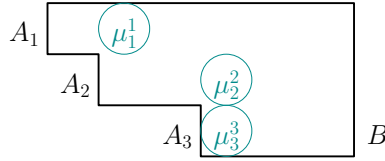


Fig. 2. One of the solutions to inequalities (9), (10) represented as points inside Young diagram. Here  $N = 4$

*Proof of Lemma 8.* Induction in  $N$ .

First, suppose that  $\lambda_{i+1} - \lambda_i \geq L/N$  for some  $1 < i < N - 1$ . Then the interlacing condition  $\mu^{N-1} \prec \mu^N = \lambda$  implies that (almost surely)  $\mu_{N-1}^{N-1} - \mu_1^{N-1} \geq L/N$ . Then we can use the induction assumption which yields the inequality (7).

If  $\lambda_{i+1} - \lambda_i < L/N$  for all  $1 < i < N - 1$ , then either  $\lambda_2 - \lambda_1 \geq L/N$  or  $\lambda_N - \lambda_{N-1} \geq L/N$ . Without loss of generality we assume the latter.

Let us fix the values of  $\mu_{j-1}^j$ ,  $j = 2, \dots, N - 1$ :

$$\mu_1^2 = A_1, \quad \mu_2^3 = A_2, \quad \dots, \mu_{N-2}^{N-1} = A_{N-2} \quad (8)$$

Clearly, if we prove the inequality (7) conditional on (8), then the same inequality would hold without conditioning.

Set also  $\lambda_{N-1} = A_{N-1}$ ,  $\lambda_N = B$ . Note that

$$A_1 \leq A_2 \leq \dots \leq A_{N-1} < B.$$

Now the distribution of  $\mu_1^1, \mu_2^2, \dots, \mu_{N-1}^{N-1}$  is uniform on the set defined by inequalities

$$\mu_1^1 \leq \mu_2^2 \leq \dots \leq \mu_{N-1}^{N-1} \leq B, \quad \mu_i^i \geq A_i, \quad i = 1, \dots, N - 1 \quad (9)$$

and also

$$\mu_i^i > A_{i-1}, \quad i = 1, \dots, N - 1. \quad (10)$$

Note that when the numbers  $A_i$  are distinct, then the inequalities (10) are automatically implied by (9). On the other hand, if  $A_i = A_{i+1} = \dots = A_{i+m}$ , then the inequalities for  $\mu_{i+1}^{i+1}, \dots, \mu_{i+m}^{i+m}$  in (9) become strict. Graphically, we can view the solutions to inequalities (9), (10) as  $N - 1$  points in  $N - 1$ -rows of a Young diagram, as shown in Fig. 2. From now on we assume that all  $A_i$  are distinct, the case of equal  $A_i$ s can be studied in the same way.

Let  $S(N - 1; A_1, \dots, A_{N-1}; B)$  denote the number of  $(N - 1)$ -tuples  $(\mu_1^1 \leq \mu_2^2 \leq \dots \leq \mu_{N-1}^{N-1})$  solving (9), (10). The definition readily implies the following monotonicity: if  $A'_i \leq A_i$ ,  $i = 1, \dots, N - 1$  and  $B' \geq B$ , then

$$S(N - 1; A_1, \dots, A_{N-1}; B) \leq S(N - 1; A'_1, \dots, A'_{N-1}; B'). \quad (11)$$

Let us prove two estimates:

$$\text{Prob} \left( \mu_1^1 \leq A_1 + \frac{B - A_1}{2^N} \right) \geq 2^{-N-1}, \quad (12)$$

$$\text{Prob} \left( \mu_1^1 \geq A_1 + \frac{B - A_1}{2^{N-1}} \right) \geq 2^{-N-1}. \quad (13)$$



These two estimates together with observation that  $B - A_1 \geq B - A_{N-1} = L$  readily imply (7).

To prove (12) note that conditionally on  $\mu_2^2, \dots, \mu_{N-1}^{N-1}$  the distribution of  $\mu_1^1$  [which arises from the uniform measure on the set defined by inequalities (9), (10)] is uniform on the interval  $\{A_1, A_1 + 1, \dots, \mu_2^2\}$ . Since  $\mu_2^2 \leq B$ , the desired inequality immediately follows.

To prove (13), observe, first, that the distribution of  $\mu_{N-1}^{N-1}$  is given by

$$\text{Prob} \left( \mu_{N-1}^{N-1} = k \right) = \frac{S(N-2; A_1, \dots, A_{N-2}; k)}{S(N-1; A_1, \dots, A_{N-1}; B)}, \quad k = A_{N-1}, A_{N-1} + 1, \dots, B. \quad (14)$$

The monotonicity property (11) implies that the probability (14) is an increasing function of  $k$ . Therefore,

$$\text{Prob} \left( \mu_{N-1}^{N-1} \geq \frac{A_{N-1} + B}{2} \right) \geq \frac{1}{2}. \quad (15)$$

Similarly studying the conditional distribution of  $\mu_{N-2}^{N-2}$  given that  $\mu_{N-1}^{N-1} = k$ , we get

$$\text{Prob} \left( \mu_{N-2}^{N-2} \geq \frac{A_{N-2} + k}{2} \mid \mu_{N-1}^{N-1} = k \right) \geq \frac{1}{2} \quad (16)$$

Combining (15) and (16) we conclude that

$$\text{Prob} \left( \mu_{N-2}^{N-2} \geq \frac{3A_{N-2} + B}{4} \right) \geq \frac{1}{2^2}. \quad (17)$$

Further studying in the same way the conditional distribution of  $\mu_{N-3}^{N-3}$  given  $\mu_{N-2}^{N-2}$  and  $\mu_{N-1}^{N-1}$  and combining with (17) we get

$$\text{Prob} \left( \mu_{N-3}^{N-3} \geq \frac{7A_{N-3} + B}{8} \right) \geq \frac{1}{8}. \quad (18)$$

Continuing this process, we finally get the inequality

$$\text{Prob} \left( \mu_1^1 \geq \frac{(2^{N-1} - 1)A_1 + B}{2^{N-1}} \right) \geq 2^{1-N}, \quad (19)$$

which is (13).  $\square$

#### 4. Proof of Theorem 1

Proposition 7 yields that centered and rescaled random variables  $\xi(N)_b^a, a = 1, \dots, k, b = 1, \dots, a$  are tight as  $N \rightarrow \infty$ . Let  $\zeta_b^a$  denote any subsequential limit of the random vectors

$$\sqrt{\frac{8}{3N}} \left( \xi(N)_b^a - N/2 \right), \quad a = 1, \dots, k, \quad b = 1, \dots, a. \quad (20)$$

Since the distribution of  $\xi(N)_b^a$  for any  $N$  satisfies the Gibbs property on  $\mathbb{GT}^{(k)}$ , the distribution of  $\zeta_b^a$  satisfies the (continuous) Gibbs property on  $\widehat{\mathbb{GT}}^{(k)}$ . Now the combination of Proposition 6 and Theorem 2 yields that the distribution of  $\zeta$  is the GUE-corners process. Since all the subsequential limits are the same, we conclude that (20) weakly converges to the GUE-corners process.

In particular, this implies that with probability tending to 1 all the coordinates of random vector  $\xi(N)_b^a$  become distinct as  $N \rightarrow \infty$ . This yields part 1 of Theorem 1. Further, when the coordinates  $\xi(N)_b^a$  are distinct, then  $\xi(N)_b^a = \eta(N)_b^a$ , which finishes the proof of Theorem 1.

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